Chapter 7:
Techniques of Integration

Section 7.3:
Trigonometric Substitution

Trigonometric Substitution

Idea: We are going to be integrating functions that involve roots with addition or subtraction in them, like…

$$\int \sqrt{1-x^2}\,dx \qquad \int \frac{x}{\sqrt{x^2-1}}\,dx \qquad \int \frac{x^3}{\sqrt{1+x^2}}\,dx$$

Roots suck!!!

We want to get rid of the roots.

We'll use a trig. sub. (turns into a trig. integral)

To decide which trig. sub. recall trig. identities

Some Trig. Identities you need to know…

$$sin^2x + cos^2x = 1$$

$$\rightarrow sin^2x = 1 - cos^2x$$

$$\rightarrow cos^2x = 1 - sin^2x$$

$$tan^2x + 1 = sec^2x$$

$$\rightarrow tan^2x = sec^2x - 1$$

$$\rightarrow sec^2x = tan^2x + 1$$

Trigonometric Substitution

Idea: $\int \sqrt{1 - x^2}\, dx$

Trig. identities …

$1 - cos^2 x = sin^2 x$

$1 - sin^2 x = cos^2 x$

$sec^2 x - 1 = tan^2 x$

$tan^2 x + 1 = sec^2\ x$

Use substitution …

Trigonometric Substitution

Idea: $\int \frac{x}{\sqrt{x^2-1}}\,dx$

Trig. identities …

$1 - cos^2 x = sin^2 x$

$1 - sin^2 x = cos^2 x$

$sec^2 x - 1 = tan^2 x$

$tan^2 x + 1 = sec^2\ x$

Use substitution …

Trigonometric Substitution

Idea: $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Trig. identities …

$1 - cos^2 x = sin^2 x$

$1 - sin^2 x = cos^2 x$

$sec^2 x - 1 = tan^2 x$

$tan^2 x + 1 = sec^2\ x$

Use substitution …

Trigonometric Substitution

Notes:

- Make sure to find dx
- Make sure that your new integral only has $\theta's$ in it
- Your final answer should only have the original variable in it (you will probably have to use the triangle trick here)

Trigonometric Substitution

Ex 1: Find $\int \sqrt{1-x^2}\,dx$

Trigonometric Substitution

Ex 2: Find $\int \frac{x}{\sqrt{x^2-1}}\,dx$

Trigonometric Substitution

Ex 3: Find $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Trigonometric Substitution

Issue #1:

Q: The number in front of the x^2 term must be a 1. What if it isn't?

A: Put an appropriate number in front of your trig sub to cancel out the number.

Ex 4: Find $\int \sqrt{1-3x^2}\,dx$

Trigonometric Substitution

Issue #2:

Q: The number being added inside the root must be a 1. What if it isn't?

A: Put an appropriate number in front of your trig sub so you can factor out the number and end up with a 1 in its place.

Ex 5: Find $\int \sqrt{5 - x^2}\, dx$

Trigonometric Substitution

Issue #1 and #2:

Q: What if the number in front of the x^2 terms isn't a 1, and the number being added or subtracted isn't a 1?

A: Put an appropriate number in front of your tirg sub so that…

1. The number in front of the x^2 cancels out and changes to the number being added or subtracted.
2. Factor out the number being added or subtracted

Trigonometric Substitution

Issue #1 and #2:

Ex 6: Find $\int \sqrt{9 + 4x^2}\, dx$

Trigonometric Substitution

Issue #3:

Q: What if besides an number and an x^2 terms, there is also an x terms inside the root?

A: Complete the square

Trigonometric Substitution

Issue #3:

A: Complete the square

1. Separate the x & x^2 terms from the number term
2. Factor out the number in front of the x^2 term
3. Complete the square on the parenthesis
4. Do I trig sub with an appropriate number in front of the trig function.

You may do something like…

$$\text{let } x + 1 = \sqrt{2}tan\theta$$

Trigonometric Substitution

Issue #1, #2, and #3:

Ex 7: Find $\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$